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Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations

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Abstract

We study optimal control problems for (time-) delayed stochastic differential equations with jumps. We establish sufficient and necessary (Pontryagin type) maximum principles for an optimal control of such systems. The associated adjoint processes are shown to satisfy a (time-) advanced backward stochastic differential equation (ABSDE). Several results on existence and uniqueness of such ABSDEs are shown. The results are illustrated by an application to optimal consumption from a cash flow with delay.

1 Introduction

Let $B(t) = B(t,\omega)$ be a Brownian motion and $\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt$, where ν is the Lévy measure of the jump measure $N(\cdot,\cdot)$, be an independent compensated Poisson random measure on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$.

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We consider a controlled stochastic delay equation of the form

$$\begin{aligned} dX(t) &= b(t,X(t),Y(t),A(t),u(t),\omega)dt + \sigma(t,X(t),Y(t),A(t),u(t),\omega)dB(t) \\ &+ \int_{\mathbb{R}} \theta(t,X(t),Y(t),A(t),u(t),z,\omega)\tilde{N}(dt,dz)\ ;\ t\in[0,T] \end{aligned} \tag{1.1}$$

$$X(t) = x_0(t)\ ;\ t\in[-\delta,0], \tag{1.2}$$

where

$$Y(t) = X(t-\delta),\quad A(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)}X(r)dr, \tag{1.3}$$

and $\delta > 0$, $\rho \geq 0$ and $T > 0$ are given constants. Here

$$\begin{aligned} b &:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{U}\times\Omega\to\mathbb{R} \\ \sigma &:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{U}\times\Omega\to\mathbb{R} \end{aligned}$$

and

$$\theta : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{U}\times\mathbb{R}_0\times\Omega\to\mathbb{R}$$

are given functions such that, for all t, $b(t,x,y,a,u,\cdot)$, $\sigma(t,x,y,a,u,\cdot)$ and $\theta(t,x,y,a,u,z,\cdot)$ are $\mathcal{F}_t$-measurable for all $x\in\mathbb{R}$, $y\in\mathbb{R}$, $a\in\mathbb{R}$, $u\in\mathcal{U}$ and $z\in\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. The function $x_0(t)$ is assumed to be continuous, deterministic.

Let $\mathcal{E}_t \subseteq \mathcal{F}_t\ ;\ t\in[0,T]$ be a given subfiltration of $\{\mathcal{F}_t\}_{t\in[0,T]}$, representing the information available to the controller who decides the value of $u(t)$ at time t. For example, we could have $\mathcal{E}_t = \mathcal{F}_{(t-c)^+}$ for some given $c > 0$. Let $\mathcal{U}\subset\mathbb{R}$ be a given set of admissible control values $u(t)\ ;\ t\in[0,T]$ and let $\mathcal{A}_{\mathcal{E}}$ be a given family of admissible control processes $u(\cdot)$, included in the set of càdlàg, $\mathcal{E}$-adapted and $\mathcal{U}$-valued processes $u(t)\ ;\ t\in[0,T]$ such that (1.1)-(1.2) has a unique solution $X(\cdot)\in L^2(\lambda\times P)$ where λ denotes the Lebesgue measure on $[0,T]$.

The performance functional is assumed to have the form

$$J(u) = E\left[\int_0^T f(t,X(t),Y(t),A(t),u(t),\omega)dt + g(X(T),\omega)\right]\ ;\ u\in\mathcal{A}_{\mathcal{E}} \tag{1.4}$$

where $f = f(t,x,y,a,u,\omega) : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{U}\times\Omega\to\mathbb{R}$ and $g = g(x,\omega) : \mathbb{R}\times\Omega\to\mathbb{R}$ are given $\mathcal{C}^1$ functions w.r.t. (x,y,a,u) such that

$$\begin{aligned} E[\int_0^T &\{|f(t,X(t),A(t),u(t))| + \left|\frac{\partial f}{\partial x_i}(t,X(t),Y(t),A(t),u(t))\right|^2\}dt \\ &+|g(X(T))| + |g'(X(T))|^2] < \infty \text{ for } x_i = x,y,a \text{ and } u. \end{aligned}$$

Here, and in the following, we suppress the ω, for notational simplicity. The problem we consider in this paper is the following:

Find $\Phi(x_0)$ and $u^*\in\mathcal{A}_{\mathcal{E}}$ such that

$$\Phi(x_0) := \sup_{u\in\mathcal{A}_{\mathcal{E}}} J(u) = J(u^*). \tag{1.5}$$

Any control $u^* \in \mathcal{A}_{\mathcal{E}}$ satisfying (1.5) is called an *optimal control*.

Variants of this problem have been studied in several papers. Stochastic control of delay systems is a challenging research area, because delay systems have, in general, an infinite-dimensional nature. Hence, the natural general approach to them is infinite-dimensional. For this kind of approach in the context of control problems we refer to [1, 7, 8, 9] in the stochastic Brownian case. To the best of our knowledge, despite the statement of a result in [19], this kind of approach was not developed for delay systems driven by a Lévy noise.

Nonetheless, in some cases still very interesting for the applications, it happens that systems with delay can be reduced to finite-dimensional systems, since the information we need from their dynamics can be represented by a finite-dimensional variable evolving in terms of itself. In such a context, the crucial point is to understand when this finite dimensional reduction of the problem is possible and/or to find conditions ensuring that. There are some papers dealing with this subject in the stochastic Brownian case: we refer to [10, 6, 12, 13, 15]. The paper [3] represents an extension of [13] to the case when the equation is driven by a Lévy noise.

We also mention the paper [5], where certain control problems of stochastic functional differential equations are studied by means of the Girsanov transformation. This approach, however, does not work if there is a delay in the noise components.

Our approach in the current paper is different from all the above. Note that the presence of the terms $Y(t)$ and $A(t)$ in (1.1) makes the problem non-Markovian and we cannot use a (finite dimensional) dynamic programming approach. However, we will show that it is possible to obtain a (Pontryagin type) maximum principle for the problem. To this end, we define the *Hamiltonian*

$$H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \to \mathbb{R}$$

by

$$\begin{aligned} H(t,x,y,a,u,p,q,r(\cdot),\omega) = H(t,x,y,a,u,p,q,r(\cdot)) = f(t,x,y,a,u) \\ + b(t,x,y,a,u)p + \sigma(t,x,y,a,u)q + \int_{\mathbb{R}_0} \theta(t,x,y,a,u,z)r(z)\nu(dz); \end{aligned} \tag{1.6}$$

where $\mathcal{R}$ is the set of functions $r : \mathbb{R}_0 \to \mathbb{R}$ such that the last term in (1.6) converges.

We assume that b, σ and θ are $\mathcal{C}^1$ functions with respect to (x,y,a,u) and that

$$\begin{aligned} E\Bigg[\int_0^T \Bigg\{ \left|\frac{\partial b}{\partial x_i}(t,X(t),Y(t),A(t),u(t))\right|^2 + \left|\frac{\partial \sigma}{\partial x_i}(t,X(t),Y(t),A(t),u(t))\right|^2 \\ + \int_{\mathbb{R}_0} \left|\frac{\partial \theta}{\partial x_i}(t,X(t),Y(t),A(t),u(t),z)\right|^2 \nu(dz) \Bigg\} dt \Bigg] < \infty \end{aligned} \tag{1.7}$$

for $x_i = x, y, a$ and u.

Associated to H we define the adjoint processes $p(t), q(t), r(t,z)$; $t \in [0,T]$, $z \in \mathbb{R}_0$, by the following backward stochastic differential equation (BSDE):

$$\begin{cases} dp(t) & = E[\mu(t)|\mathcal{F}_t]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) \ ; \ t \in [0,T] \\ p(T) & = g'(X(T)), \end{cases} \tag{1.8}$$

where

$$\begin{aligned} \mu(t) = & -\frac{\partial H}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t,\cdot)) \\ & - \frac{\partial H}{\partial y}(t+\delta, X(t+\delta), Y(t+\delta), A(t+\delta), u(t+\delta), p(t+\delta), q(t+\delta), r(t+\delta,\cdot))\chi_{[0,T-\delta]}(t) \\ & - e^{\rho t}\left(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s, X(s), Y(s), A(s), u(s), p(s), q(s), r(s,\cdot))e^{-\rho s}\chi_{[0,T]}(s)ds\right). \end{aligned} \tag{1.9}$$

Note that this BSDE is *anticipative*, or *time-advanced* in the sense that the driver $\mu(t)$ contains future values of $X(s), u(s), p(s), q(s), r(s,\cdot)$; $s \leq t+\delta$.

In the case when there are no jumps and no integral term in (1.9), anticipative BSDEs (ABSDEs for short) have been studied by [18], who prove existence and uniqueness of such equations under certain conditions. They also relate a class of linear ABSDEs to a class of linear stochastic delay control problems where there is no delay in the noise coefficients. Thus, in our paper we extend this relation to general nonlinear control problems and general nonlinear ABSDEs by means of the maximum principle, where we throughout the study include the possibility of delays also in the noise coefficients, as well as the possibility of jumps.

2 A sufficient maximum principle

In this section we establish a maximum principe of sufficient type, i.e. we show that -under some assumptions- maximizing the Hamiltonian leads to an optimal control.

Theorem 2.1 (Sufficient maximum principle) *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding state processes $\hat{X}(t), \hat{Y}(t), \hat{A}(t)$ and adjoint processes $\hat{p}(t), \hat{q}(t), \hat{r}(t,z)$, assumed to satisfy the ABSDE* (1.8)-(1.9)*. Suppose the following hold:*

(i) *The functions $x \to g(x)$ and*

$$(x,y,a,u) \to H(t,x,y,a,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \tag{2.1}$$

are concave, for each $t \in [0,T]$, a.s.

(ii)

$$E\left[\int_0^T \left\{\hat{p}(t)^2\left(\sigma^2(t)+\int_{\mathbb{R}_0}\theta^2(t,z)\nu(dz)\right)\right.\right.$$
$$\left.\left.+X^2(t)\left(\hat{q}^2(t)+\int_{\mathbb{R}_0}\hat{r}^2(t,z)\nu(dz)\right)\right\}dt\right]<\infty \tag{2.2}$$

for all $u \in \mathcal{A}_{\mathcal{E}}$.

(iii)

$$\max_{v\in\mathcal{U}} E\left[H(t,\hat{X}(t),\hat{X}(t-\delta),\hat{A}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\mid\mathcal{E}_t\right]$$
$$=E\left[H(t,\hat{X}(t),\hat{X}(t-\delta),\hat{A}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\mid\mathcal{E}_t\right] \tag{2.3}$$

for all $t\in[0,T]$, *a.s.*

Then $\hat{u}(t)$ *is an optimal control for the problem* (1.5).

Proof. Choose $u\in\mathcal{A}_{\mathcal{E}}$ and consider

$$J(u)-J(\hat{u})=I_1+I_2 \tag{2.4}$$

where

$$I_1=E\left[\int_0^T\{f(t,X(t),Y(t),A(t),u(t))-f(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{u}(t))\}dt\right] \tag{2.5}$$

$$I_2=E[g(X(T))-g(\hat{X}(T))]. \tag{2.6}$$

By the definition of H and concavity of H we have

$$\begin{aligned}
I_1 &= E\left[\int_0^T\{H(t,X(t),Y(t),A(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\right.\\
&\quad -H(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\\
&\quad -(b(t,X(t),Y(t),A(t),u(t))-b(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{u}(t)))\hat{p}(t)\\
&\quad -(\sigma(t,X(t),Y(t),A(t),u(t))-\sigma(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{u}(t)))\hat{q}(t)\\
&\quad \left.-\int_{\mathbb{R}}(\theta(t,X(t),Y(t),A(t),u(t),z)-\theta(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{u}(t),z))\hat{r}(t,z)\nu(dz)\}dt\right]\\
&\le E\left[\int_0^T\{\frac{\partial\hat{H}}{\partial x}(t)(X(t)-\hat{X}(t))+\frac{\partial\hat{H}}{\partial y}(t)(Y(t)-\hat{Y}(t))+\frac{\partial\hat{H}}{\partial a}(t)(A(t)-\hat{A}(t))\right.\\
&\quad +\frac{\partial H}{\partial u}(t)(u(t)-\hat{u}(t))-(b(t)-\hat{b}(t))\hat{p}(t)-(\sigma(t)-\hat{\sigma}(t))\hat{q}(t)\\
&\quad \left.-\int_{\mathbb{R}}(\theta(t,z)-\hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)\}dt\right],
\end{aligned} \tag{2.7}$$

where we have used the abbreviated notation

$$\begin{aligned}
\frac{\partial \hat{H}}{\partial x}(t) &= \frac{\partial H}{\partial x}(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)),\\
b(t) &= b(t, X(t), Y(t), A(t), u(t)),\\
\hat{b}(t) &= b(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t) \text{ etc.}
\end{aligned}$$

Since g is concave we have, by (2.2),

$$\begin{aligned}
I_2 &\le E[g'(\hat{X}(T))(X(T) - \hat{X}(T))] = E[\hat{p}(T)(X(T) - \hat{X}(T))]\\
&= E\left[\int_0^T \hat{p}(t)(dX(t) - d\hat{X}(t)) + \int_0^T (X(t) - \hat{X}(t))d\hat{p}(t)\right.\\
&\quad \left. + \int_0^T (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\right]\\
&= E\left[\int_0^T (b(t) - \hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t) - \hat{X}(t))E[\mu(t)|\mathcal{F}_t]dt\right.\\
&\quad \left. + \int_0^T (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\right]. \qquad (2.8)
\end{aligned}$$

Combining (2.4)-(2.8) we get, using that $X(t) = \hat{X}(t) = x_0(t)$ for all $t \in [-\delta, 0]$,

$$\begin{aligned}
J(u) - J(\hat{u}) &\le E\left[\int_0^T \left\{\frac{\partial H}{\partial x}(t)(X(t) - \hat{X}(t)) + \frac{\partial H}{\partial y}(t)(Y(t) - \hat{Y}(t))\right.\right.\\
&\quad \left.\left. + \frac{\partial \hat{H}}{\partial a}(t)(A(t) - \hat{A}(t)) + \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) + \mu(t)(X(t) - \hat{X}(t))\right\} dt\right]\\
&= E\left[\int_\delta^{T+\delta} \left\{\frac{\partial \hat{H}}{\partial x}(t-\delta) + \frac{\partial \hat{H}}{\partial y}(t)\chi_{[0,T]}(t) + \mu(t-\delta)\right\}(Y(t) - \hat{Y}(t))dt\right.\\
&\quad \left. + \int_0^T \frac{\partial \hat{H}}{\partial a}(t)(A(t) - \hat{A}(t))dt + \int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))dt\right]. \qquad (2.9)
\end{aligned}$$

Using integration by parts and substituting $r = t - \delta$, we get

$$\begin{aligned}
\int_0^T \frac{\partial \hat{H}}{\partial a}(s)(A(s) - \hat{A}(s))ds &= \int_0^T \frac{\partial \hat{H}}{\partial a}(s)\int_{s-\delta}^s e^{-\rho(s-r)}(X(r) - \hat{X}(r))drds\\
&= \int_0^T \left(\int_r^{r+\delta} \frac{\partial \hat{H}}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s)ds\right)e^{\rho r}(X(r) - \hat{X}(r))dr\\
&= \int_\delta^{T+\delta} \left(\int_{t-\delta}^t \frac{\partial \hat{H}}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s)ds\right)e^{\rho(t-\delta)}(X(t-\delta) - \hat{X}(t-\delta))dt. \qquad (2.10)
\end{aligned}$$

Combining this with (2.9) and using (1.9) we obtain

$$\begin{aligned}
J(u) - J(\hat{u}) \leq & \left[\int_{\delta}^{T+\delta} \left\{ \frac{\partial \hat{H}}{\partial x}(t-\delta) + \frac{\partial \hat{H}}{\partial y}(t)\chi_{[0,T]}(t) \right.\right. \\
& + \left. \left(\int_{t-\delta}^{t} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} \chi_{[0,T]}(s) ds \right) e^{\rho(t-\delta)} + \mu(t-\delta) \right\} (Y(t) - \hat{Y}(t)) dt \\
& + \left. \int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) dt \right] \\
= & \, E\left[\int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) dt \right] \\
= & \, E\left[\int_0^T E\left[\frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) \mid \mathcal{E}_t \right] dt \right] \\
= & \, E\left[\int_0^T E\left[\frac{\partial \hat{H}}{\partial u}(t) \mid \mathcal{E}_t \right] (u(t) - \hat{u}(t)) dt \right] \leq 0.
\end{aligned}$$

The last inequality holds because $v = \hat{u}(t)$ maximizes $E[H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot) \mid \mathcal{E}_t]$ for each $t \in [0,T]$. This proves that $\hat{u}$ is an optimal control. $\square$

3 A necessary maximum principle

A drawback with the sufficient maximum principle in Section 2 is the condition of concavity, which does not always hold in the applications. In this section we will prove a result going in the other direction. More precisely, we will prove the equivalence between being a directional critical point for $J(u)$ and a critical point for the conditional Hamiltonian. To this end, we need to make the following assumptions:

A 1 *For all $u \in \mathcal{A}_{\mathcal{E}}$ and all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ there exists $\varepsilon > 0$ such that*

$$u + s\beta \in \mathcal{A}_{\mathcal{E}} \text{ for all } s \in (-\varepsilon, \varepsilon).$$

A 2 *For all $t_0 \in [0,T]$ and all bounded $\mathcal{E}_{t_0}$-measurable random variables α the control process $\beta(t)$ defined by*

$$\beta(t) = \alpha \chi_{[t_0,T]}(t) \; ; \; t \in [0,T] \tag{3.1}$$

belongs to $\mathcal{A}_{\mathcal{E}}$.

A 3 *For all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ the derivative process*

$$\xi(t) := \frac{d}{ds} X^{u+s\beta}(t) \mid_{s=0} \tag{3.2}$$

exists and belongs to $L^2(\lambda \times P)$.

It follows from (1.1) that

$$
\begin{aligned}
d\xi(t) = &\left\{ \frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t) \right\} dt \\
&+ \left\{ \frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \sigma}{\partial u}(t)\beta(t) \right\} dB(t) \\
&+ \int_{\mathbb{R}_0} \left\{ \frac{\partial \theta}{\partial x}(t,z)\xi(t) + \frac{\partial \theta}{\partial y}(t,z)\xi(t-\delta) \right. \\
&\left. + \frac{\partial \theta}{\partial a}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \theta}{\partial u}(t)\beta(t) \right\} \tilde{N}(dt,dz) \qquad (3.3)
\end{aligned}
$$

where we for simplicity of notation have put

$$
\frac{\partial b}{\partial x}(t) = \frac{\partial b}{\partial x}(t, X(t), X(t-\delta), A(t), u(t)) \text{ etc } \ldots
$$

and we have used that

$$
\frac{d}{ds} Y^{u+s\beta}(t) \mid_{s=0} = \frac{d}{ds} X^{u+s\beta}(t-\delta) \mid_{s=0} = \xi(t-\delta) \qquad (3.4)
$$

and

$$
\begin{aligned}
\frac{d}{ds} A^{u+s\beta}(t) \mid_{s=0} &= \frac{d}{ds} \left(\int_{t-\delta}^{t} e^{-\rho(t-r)} X^{u+s\beta}(r)dr \right) \mid_{s=0} \\
&= \int_{t-\delta}^{t} e^{-\rho(t-r)} \frac{d}{ds} X^{u+s\beta}(r) \mid_{s=0} dt = \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr. \qquad (3.5)
\end{aligned}
$$

Note that

$$
\xi(t) = 0 \text{ for } t \in [-\delta, 0]. \qquad (3.6)
$$

Theorem 3.1 (Necessary maximum principle) *Suppose $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{X}(t)$ of* (1.1)-(1.2) *and $\hat{p}(t)$, $\hat{q}(t)$, $\hat{r}(t,z)$ of* (1.7)-(1.8) *and corresponding derivative process $\hat{\xi}(t)$ given by* (3.2).

Assume that

$$
\begin{aligned}
E\Bigg[&\int_0^T \hat{p}^2(t) \left\{ \left(\frac{\partial \sigma}{\partial x}\right)^2 (t)\hat{\xi}^2(t) + \left(\frac{\partial \sigma}{\partial y}\right)^2 (t)\xi^2(t-\delta) \right. \\
&+ \left(\frac{\partial \sigma}{\partial a}\right)^2 (t) \left(\int_{t-\delta}^{t} e^{-\rho(t-r)}\hat{\xi}(r)dr \right)^2 + \left(\frac{\partial \sigma}{\partial u}\right)^2 (t) \\
&+ \int_{\mathbb{R}_0} \left\{ \left(\frac{\partial \theta}{\partial x}\right)^2 (t,z)\hat{\xi}^2(t) + \left(\frac{\partial \theta}{\partial y}\right)^2 (t,z)\hat{\xi}^2(t-\delta) \right. \\
&\left.\left. + \left(\frac{\partial \theta}{\partial a}\right)^2 (t,z) \left(\int_{t-\delta}^{t} e^{-\rho(t-r)}\hat{\xi}(r)dr \right)^2 + \left(\frac{\partial \theta}{\partial u}\right)^2 (t,z) \right\} \nu(dz) \right\} dt \\
&+ \int_0^T \hat{\xi}^2(t) \left\{ \hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t,z)\nu(dz) \right\} dt \Bigg] < \infty. \qquad (3.7)
\end{aligned}
$$

Then the following are equivalent:

(i) $\frac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0}=0$ *for all bounded* $\beta \in \mathcal{A}_{\mathcal{E}}$.

(ii) $E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t\right]_{u=\hat{u}(t)} = 0$ *a.s. for all* $t \in [0,T]$.

Proof. For simplicity of notation we write $\hat{u}=u$, $\hat{X}=X$, $\hat{p}=p$, $\hat{q}=q$ and $\hat{r}=r$ in the following. Suppose **(i)** holds. Then

$$
\begin{aligned}
0 &= \frac{d}{ds}J(u+s\beta)\mid_{s=0} \\
&= \frac{d}{ds}E\left[\int_0^T f(t,X^{u+s\beta}(t),Y^{u+s\beta}(t),A^{u+s\beta}(t),u(t)+s\beta(t))dt + g(X^{u+s\beta}(T))\right]\mid_{s=0} \\
&= E\left[\int_0^T \left\{\frac{\partial f}{\partial x}(t)\xi(t)+\frac{\partial f}{\partial y}(t)\xi(t-\delta)+\frac{\partial f}{\partial a}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dt + \frac{\partial f}{\partial u}(t)\beta(t)\right\}dt + g'(X(T))\xi(T)\right] \\
&= E\left[\int_0^T \left\{\frac{\partial H}{\partial x}(t)-\frac{\partial b}{\partial x}(t)p(t)-\frac{\partial \sigma}{\partial x}(t)q(t)-\int_{\mathbb{R}}\frac{\partial \theta}{\partial x}(t,z)r(t,z)\nu(dz)\right\}\xi(t)dt\right. \\
&\quad + \int_0^T \left\{\frac{\partial H}{\partial y}(t)-\frac{\partial b}{\partial y}(t)p(t)-\frac{\partial \sigma}{\partial y}(t)q(t)-\int_{\mathbb{R}}\frac{\partial \theta}{\partial y}(t,z)r(t,z)\nu(dz)\right\}\xi(t-\delta)dt \\
&\quad + \int_0^T \left\{\frac{\partial H}{\partial a}(t)-\frac{\partial b}{\partial a}(t)p(t)-\frac{\partial \sigma}{\partial a}(t)q(t)-\int_{\mathbb{R}}\frac{\partial \theta}{\partial a}(t,z)r(t,z)\nu(dz)\right\}\left(\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr\right)dt \\
&\quad \left. + \int_0^T \frac{\partial f}{\partial u}(t)\beta(t)dt + g'(X(T))\xi(T)\right]. \qquad (3.8)
\end{aligned}
$$

By (3.3)

$$
\begin{aligned}
E[g'(X(T))\xi(T)] &= E[p(T)\xi(T)] = E\Bigg[\int_0^T p(t)d\xi(t) + \int_0^T \xi(t)dp(t) \\
&\quad + \int_0^T q(t)\left\{\frac{\partial\sigma}{\partial x}(t)\xi(t) + \frac{\partial\sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial\sigma}{\partial a}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial\sigma}{\partial u}(t)\beta(t)\right\}dt \\
&\quad + \int_0^T \int_{\mathbb{R}} r(t,z)\left\{\frac{\partial\theta}{\partial x}(t,z)\xi(t) + \frac{\partial\theta}{\partial y}(t,z)\xi(t-\delta) + \frac{\partial\theta}{\partial a}(t,z)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr \right. \\
&\qquad \left. + \frac{\partial\theta}{\partial u}(t)\beta(t)\right\}\nu(dz)dt\Bigg] \\
&= E\Bigg[\int_0^T p(t)\left\{\frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t)\right\}dt \\
&\quad + \int_0^T \xi(t)E[\mu(t)|\mathcal{F}_t]dt \\
&\quad + \int_0^T q(t)\left\{\frac{\partial\sigma}{\partial x}(t)\xi(t) + \frac{\partial\sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial\sigma}{\partial a}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial\sigma}{\partial u}(t)\beta(t)\right\}dt \\
&\quad + \int_0^T \int_{\mathbb{R}} r(t,z)\left\{\frac{\partial\theta}{\partial x}(t,z)\xi(t) + \frac{\partial\theta}{\partial y}(t,z)\xi(t-\delta) + \frac{\partial\theta}{\partial a}(t,z)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr \right. \\
&\qquad \left. + \frac{\partial\theta}{\partial u}(t,z)\beta(t)\right\}\nu(dz)dt\Bigg]
\end{aligned}
\tag{3.9}
$$

Combining (3.8) and (3.9) we get

$$
\begin{aligned}
0 &= E\left[\int_0^T \xi(t)\left\{\frac{\partial H}{\partial x}(t)+\mu(t)\right\}dt+\int_0^T \xi(t-\delta)\frac{\partial H}{\partial y}(t)dt\right.\\
&\quad+\left.\int_0^T\left(\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr\right)\frac{\partial H}{\partial a}(t)dt+\int_0^T\frac{\partial H}{\partial u}(t)\beta(t)dt\right]\\
&= E\left[\int_0^T \xi(t)\left\{\frac{\partial H}{\partial x}(t)-\frac{\partial H}{\partial x}(t)-\frac{\partial H}{\partial y}(t+\delta)\chi_{[0,T-\delta]}(t)\right.\right.\\
&\quad\left.-e^{\rho t}\left(\int_t^{t+\delta}\frac{\partial H}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s)ds\right)\right\}dt+\int_0^T\xi(t-\delta)\frac{\partial H}{\partial y}(t)dt\\
&\quad+\left.\int_0^T\left(\int_{s-\delta}^s e^{-\rho(s-t)}\xi(t)dt\right)\frac{\partial H}{\partial a}(s)ds+\int_0^T\frac{\partial H}{\partial u}(t)\beta(t)dt\right]\\
&= E\left[\int_0^T\xi(t)\left\{-\frac{\partial H}{\partial y}(t+\delta)\chi_{[0,T-\delta]}(t)-e^{\rho t}\left(\int_t^{t+\delta}\frac{\partial H}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s)ds\right)\right\}dt\right.\\
&\quad+\int_0^T\xi(t-\delta)\frac{\partial H}{\partial y}(t)dt\\
&\quad+\left.e^{\rho t}\int_0^T\left(\int_t^{t+\delta}\frac{\partial H}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s)ds\right)\xi(t)dt+\int_0^T\frac{\partial H}{\partial u}(t)\beta(t)dt\right]\\
&= E\left[\int_0^T\frac{\partial H}{\partial u}(t)\beta(t)dt\right], \qquad (3.10)
\end{aligned}
$$

where we again have used integration by parts.

If we apply (3.10) to

$$\beta(t)=\alpha(\omega)\chi_{[s,T]}(t)$$

where $\alpha(\omega)$ bounded and $\mathcal{E}_{t_0}$-measurable, $s\geq t_0$, we get

$$E\left[\int_s^T\frac{\partial H}{\partial u}(t)dt\ \alpha\right]=0.$$

Differentiating with respect to s we obtain

$$E\left[\frac{\partial H}{\partial u}(s)\alpha\right]=0.$$

Since this holds for all $s\geq t_0$ and all α we conclude that

$$E\left[\frac{\partial H}{\partial u}(t_0)\mid\mathcal{E}_{t_0}\right]=0.$$

This shows that **(i)** $\Rightarrow$ **(ii)**.

Conversely, since every bounded $\beta\in\mathcal{A}_{\mathcal{E}}$ can be approximated by linear combinations of controls β of the form (3.1), we can prove that **(ii)** $\Rightarrow$ **(i)** by reversing the above argument. □

4 Time-advanced BSDEs with jumps

We now study time-advanced backward stochastic differential equations driven both by Brownian motion $B(t)$ and compensated Poisson random measures $\tilde{N}(dt, dz)$.

4.1 Framework

Given a positive constant δ, denote by $D([0,\delta],\mathbb{R})$ the space of all càdlàg paths from $[0,\delta]$ into $\mathbb{R}$. For a path $X(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, X_t will denote the function defined by $X_t(s) = X(t+s)$ for $s \in [0,\delta]$. Put $\mathcal{H} = L^2(\nu)$. Consider the L^2 spaces $V_1 := L^2([0,\delta], ds)$ and $V_2 := L^2([0,\delta] \to \mathcal{H}, ds)$. Let

$$F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times V_1 \times \mathbb{R} \times \mathbb{R} \times V_1 \times \mathcal{H} \times \mathcal{H} \times V_2 \times \Omega \to \mathbb{R}$$

be a predictable function. Introduce the following Lipschitz condition: There exists a constant C such that

$$\begin{aligned}
&|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r,\omega) - F(t,\bar{p}_1,\bar{p}_2,\bar{p},\bar{q}_1,\bar{q}_2,\bar{q},\bar{r}_1,\bar{r}_2,\bar{r},\omega)| \\
&\quad \le C(|p_1-\bar{p}_1| + |p_2-\bar{p}_2| + |p-\bar{p}|_{V_1} + |q_1-\bar{q}_1| + |q_2-\bar{q}_2| + |q-\bar{q}|_{V_1} \\
&\quad + |r_1-\bar{r}_1|_{\mathcal{H}} + |r_2-\bar{r}_2|_{\mathcal{H}} + |r-\bar{r}|_{V_2}.
\end{aligned} \tag{4.1}$$

4.2 First existence and uniqueness theorem

We first consider the following time-advanced backward stochastic differential equation in the unknown $\mathcal{F}_t$ adapted processes $(p(t), q(t), r(t,z))$:

$$\begin{aligned}
dp(t) = E[F\big(&t, p(t), p(t+\delta)\chi_{[0,T-\delta]}(t), p_t\chi_{[0,T-\delta]}(t), q(t), q(t+\delta)\chi_{[0,T-\delta]}(t), \\
&q_t\chi_{[0,T-\delta]}(t), r(t), r(t+\delta)\chi_{[0,T-\delta]}(t), r_t\chi_{[0,T-\delta]}(t)\big)\,|\mathcal{F}_t]dt \\
&+ q(t)dB(t) + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz)\ ;\ t \in [0,T]
\end{aligned} \tag{4.2}$$

$$p(T) = G, \tag{4.3}$$

where G is a given $\mathcal{F}_T$-measurable random variable.

Note that the time-advanced BSDE (1.8)-(1.9) for the adjoint processes of the Hamiltonian is of this form.

For this type of time-advanced BSDEs we have the following result:

Theorem 4.1 *Assume that $E[G^2] < \infty$ and that condition* (4.1) *is satisfied. Then the BSDE* (4.2)-(4.3) *has a unique solution $p(t), q(t), r(t,z))$ such that*

$$E\left[\int_0^T \left\{p^2(t) + q^2(t) + \int_{\mathbb{R}} r^2(t,z)\nu(dz)\right\} dt\right] < \infty. \tag{4.4}$$

Moreover, the solution can be found by inductively solving a sequence of BSDEs backwards as follows:

Step 0: In the interval $[T-\delta, T]$ we let $p(t), q(t)$ and $r(t,z)$ be defined as the solution of the classical BSDE

$$
\begin{aligned}
dp(t) &= F\left(t, p(t), 0, 0, q(t), 0, 0, r(t,z), 0, 0\right) dt \\
&\quad + q(t)dB(t) + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz) \ ; \ t \in [T-\delta, T] \qquad (4.5) \\
p(T) &= G. \qquad (4.6)
\end{aligned}
$$

Step k ; $k \geq 1$**:** If the values of $(p(t), q(t), r(t,z))$ have been found for $t \in [T-k\delta, T-(k-1)\delta]$, then if $t \in [T-(k+1)\delta, T-k\delta]$ the values of $p(t+\delta), p_t, q(t+\delta), q_t, r(t+\delta, z)$ and r_t are known and hence the BSDE

$$
\begin{aligned}
dp(t) &= E[F\left(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t\right) |\mathcal{F}_t]dt \\
&\quad + q(t)dB(t) + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz) \ ; \ t \in [T-(k+1)\delta, T-k\delta] \qquad (4.7) \\
p(T-k\delta) &= \text{ the value found in Step } k-1 \qquad (4.8)
\end{aligned}
$$

has a unique solution in $[T-(k+1)\delta, T-k\delta]$.

We proceed like this until k is such that $T-(k+1)\delta \leq 0 < T-k\delta$ and then we solve the corresponding BSDE on the interval $[0, T-k\delta]$.

Proof. The proof follows directly from the above inductive procedure. The estimate (4.4) is a consequence of known estimates for classical BSDEs. □

4.3 Second existence and uniqueness theorem

Next, we consider the following backward stochastic differential equation in the unknown $\mathcal{F}_t$-adapted processes $(p(t), q(t), r(t,x))$:

$$
\begin{aligned}
dp(t) &= E[F(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t)|\mathcal{F}_t]dt \\
&\quad + q(t)dB_t + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz), \ ; \ t \in [0,T] \qquad (4.9) \\
p(t) &= G(t), \qquad t \in [T, T+\delta]. \qquad (4.10)
\end{aligned}
$$

where G is a given continuous $\mathcal{F}_t$-adapted stochastic process.

Theorem 4.2 *Assume* $E[\sup_{T \leq t \leq T+\delta} |G(t)|^2] < \infty$ *and that the condition* (4.1) *is satisfied. Then the backward stochastic differential equation* (4.9) *admits a unique solution* $(p(t), q(t), r(t,z))$ *such that*

$$
E[\int_0^T \{p^2(t) + q^2(t) + \int_{\mathbb{R}} r^2(t,z)\nu(dz)\}dt] < \infty.
$$

Proof.

Step 1 Assume F is independent of p_1, p_2 and p. Set $q^0(t) := 0, r^0(t,x) = 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t,x))$ to be the unique solution to the following backward stochastic differential equation equation:

$$\begin{aligned} dp^n(t) = E[F(t, q^{n-1}(t), q^{n-1}(t+\delta), q_t^{n-1}, r^{n-1}(t,\cdot), r^{n-1}(t+\delta,\cdot), r_t^{n-1}(\cdot))|\mathcal{F}_t]dt \\ + q^n(t)dB_t + r^n(t,z)\tilde{N}(dt,dz), \quad t \in [0,T] \end{aligned} \tag{4.11}$$

$$p^n(t) = G(t) \qquad t \in [T, T+\delta].$$

It is a consequence of the martingale representation theorem that the above equation admits a unique solution, see, e.g. [22], [17]. We extend q^n, r^n to $[0, T+\delta]$ by setting $q^n(s) = 0$, $r^n(s,z) = 0$ for $T \leq s \leq T+\delta$. We are going to show that $(p^n(t), q^n(t), r^n(t,x))$ forms a Cauchy sequence. By Itô's formula, we have

$$\begin{aligned} 0 &= |p^{n+1}(T) - p^n(T)|^2 = |p^{n+1}(t) - p^n(t)|^2 \\ &+ 2\int_t^T (p^{n+1}(s) - p^n(s))(E[F(s, q^n(s), q^n(s+\delta), q_s^n, r^n(s,\cdot), r^n(s+\delta,\cdot), r_s^n(\cdot))|\mathcal{F}_s] \\ &\quad - E[F(s, q^{n-1}(s), q^{n-1}(s+\delta), q_s^{n-1}, r^{n-1}(s,\cdot), r^{n-1}(s+\delta,\cdot), r_s^{n-1}(\cdot)))|\mathcal{F}_s]ds \\ &+ \int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|^2 ds\, \nu(dz) + \int_t^T |q^{n+1}(s) - q^n(s)|^2 ds \\ &+ 2\int_t^T (p^{n+1}(s) - p^n(s))(q^{n+1}(s) - q^n(s))dB_s \\ &+ \int_t^T \int_{\mathbb{R}} \{|r^{n+1}(s,z) - r^n(s,z)|^2 + 2(p^{n+1}(s-) - p^n(s-))(r^{n+1}(s,z) - r^n(s,z))\}\tilde{N}(ds,dz) \end{aligned} \tag{4.12}$$

Rearranging terms, in view of (4.1), we get

$$
\begin{aligned}
&E[|p^{n+1}(t)-p^n(t)|^2] \\
&\quad + E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|^2 ds\nu(dz)\right] + E\left[\int_t^T |q^{n+1}(s)-q^n(s)|^2 ds\right] \\
&\quad \le 2E\left[\int_t^T |(p^{n+1}(s)-p^n(s))(E[F(s,q^n(s),q^n(s+\delta),r^n(s,\cdot),r^n(s+\delta,\cdot))\right. \\
&\quad -F(s,q^{n-1}(s),q^{n-1}(s+\delta),r^{n-1}(s,\cdot),r^{n-1}(s+\delta,\cdot))|\mathcal{F}_s])|ds] \\
&\quad \le C_\varepsilon E\left[\int_t^T |p^{n+1}(s)-p^n(s)|^2 ds\right] + \varepsilon E\left[\int_t^T |q^n(s)-q^{n-1}(s)|^2 ds\right] \\
&\quad + \varepsilon E\left[\int_t^T |q^n(s+\delta)-q^{n-1}(s+\delta)|^2 ds\right] + \varepsilon E\left[\int_t^T (\int_s^{s+\delta}|q^n(u)-q^{n-1}(u)|^2 du) ds\right] \\
&\quad + \varepsilon E\left[\int_t^T |r^n(s)-r^{n-1}(s)|^2_{\mathcal{H}} ds\right] \\
&\quad + \varepsilon E\left[\int_t^T |r^n(s+\delta)-r^{n-1}(s+\delta)|^2_{\mathcal{H}} ds\right] + \varepsilon E\left[\int_t^T \left(\int_s^{s+\delta}|r^n(u)-r^{n-1}(u)|^2_{\mathcal{H}} du\right) ds\right]
\end{aligned}
\tag{4.13}
$$

Note that

$$
E\left[\int_t^T |q^n(s+\delta)-q^{n-1}(s+\delta)|^2 ds\right] \le E\left[\int_t^T |q^n(s)-q^{n-1}(s)|^2 ds\right]. \tag{4.14}
$$

Interchanging the order of integration,

$$
\begin{aligned}
&E\left[\int_t^T \left(\int_s^{s+\delta}|q^n(u)-q^{n-1}(u)|^2 du\right) ds\right] = E\left[\int_t^{T+\delta}|q^n(u)-q^{n-1}(u)|^2 du(\int_{u-\delta}^u ds\right] \\
&\quad \le \delta E\left[\int_t^T |q^n(s)-q^{n-1}(s)|^2 ds\right].
\end{aligned}
\tag{4.15}
$$

Similar inequalities hold also for $r^n - r^{n-1}$. It follows from (4.13) that

$$
\begin{aligned}
&E[|p^{n+1}(t)-p^n(t)|^2] \\
&\quad + E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|^2 ds\nu(dz)\right] + E\left[\int_t^T |q^{n+1}(s)-q^n(s)|^2 ds\right] \\
&\quad \le C_\varepsilon E\left[\int_t^T |p^{n+1}(s)-p^n(s)|^2 ds\right] + (2+M)\varepsilon E\left[\int_t^T |q^n(s)-q^{n-1}(s)|^2 ds\right] \\
&\quad + 3\varepsilon E\left[\int_t^T |r^n(s)-r^{n-1}(s)|^2_{\mathcal{H}} ds\right].
\end{aligned}
\tag{4.16}
$$

Choose $\varepsilon > 0$ sufficiently small so that

$$\begin{aligned}
&E[|p^{n+1}(t) - p^n(t)|^2] \\
&\quad + E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|^2 ds\ \nu(dz)\right] + E\left[\int_t^T |q^{n+1}(s) - q^n(s)|^2 ds\right] \\
&\le C_\varepsilon E\left[\int_t^T |p^{n+1}(s) - p^n(s)|^2 ds\right] + \frac{1}{2} E\left[\int_t^T |q^n(s) - q^{n-1}(s)|^2 ds\right] \\
&+ \frac{1}{2} E\left[\int_t^T |r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right].
\end{aligned} \tag{4.17}$$

This implies that

$$\begin{aligned}
&-\frac{d}{dt}\left(e^{C_\varepsilon t} E\left[\int_t^T |p^{n+1}(s) - p^n(s)|^2 ds\right]\right) \\
&\quad + e^{C_\varepsilon t} E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|^2 ds\ \nu(dz)\right] + e^{C_\varepsilon t} E\left[\int_t^T |q^{n+1}(s) - q^n(s)|^2 ds\right] \\
&\le \frac{1}{2} e^{C_\varepsilon t} E\left[\int_t^T |q^n(s) - q^{n-1}(s)|^2 ds\right] + \frac{1}{2} e^{C_\varepsilon t} E\left[\int_t^T |r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right].
\end{aligned} \tag{4.18}$$

Integrating the last inequality we get

$$\begin{aligned}
&E\left[\int_0^T |p^{n+1}(s) - p^n(s)|^2 ds\right] + \int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T |q^{n+1}(s) - q^n(s)|^2 ds\right] \\
&+ \int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|^2 ds \nu(dz)\right] \\
&\le \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T |q^n(s) - q^{n-1}(s)|^2 ds\right] + \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T |r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right]
\end{aligned} \tag{4.19}$$

In particular,

$$\begin{aligned}
&\int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|^2 ds \nu(dz)\right] + \int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T |q^{n+1}(s) - q^n(s)|^2 ds\right] \\
&\quad \le \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T |q^n(s) - q^{n-1}(s)|^2 ds\right] + \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T |r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right]
\end{aligned} \tag{4.20}$$

This yields

$$\begin{aligned}
&\int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|^2 ds \nu(dz)\right] + \int_0^T dt\ e^{C_\varepsilon t} E\left[\int_t^T |q^{n+1}(s) - q^n(s)|^2 ds\right] \\
&\quad \le \left(\frac{1}{2}\right)^n C
\end{aligned} \tag{4.21}$$

for some constant C. It follows from (4.19) that

$$E\left[\int_0^T |p^{n+1}(s)-p^n(s)|^2 ds\right] \le \left(\frac{1}{2}\right)^n C. \tag{4.22}$$

(4.16) and ((4.19) further gives

$$E\left[\int_0^T \int_{\mathbb{R}} |r^{n+1}(s,z)-r^n(s,z)|^2 ds\nu(dz)\right] + E\left[\int_0^T |q^{n+1}(s)-q^n(s)|^2 ds\right] \le \left(\frac{1}{2}\right)^n CnC_\varepsilon. \tag{4.23}$$

In view of (4.16), (4.19) and (4.20), we conclude that there exist progressively measurable processes $(p(t), q(t), r(t,z))$ such that

$$\begin{aligned}
&\lim_{n\to\infty} E[|p^n(t)-p(t)|^2] = 0,\\
&\lim_{n\to\infty} \int_0^T E[|p^n(t)-p(t)|^2]dt = 0,\\
&\lim_{n\to\infty} \int_0^T E[|q^n(t)-q(t)|^2]dt = 0,\\
&\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}} E[|r^n(t,z)-r(t,z)|^2]\nu(dz)dt = 0.
\end{aligned}$$

Letting $n\to\infty$ in (4.11) we see that $(p(t), q(t), r(t,z))$ satisfies

$$\begin{aligned}
p(t) &+ \int_t^T E[F(s, q(s), q(s+\delta), q_s, r(s,\cdot), r(s+\delta,\cdot), r_s(\cdot))|\mathcal{F}_s]ds\\
&+ \int_t^T q(s)dB_s + \int_t^T \int_{\mathbb{R}} r(s,z)\tilde{N}(ds,dz) = g(T)
\end{aligned} \tag{4.24}$$

i.e., $(p(t), q(t), r(t,z))$ is a solution. Uniqueness follows easily from the Ito's formula, a similar calculation of deducing (4.12) and (4.13), and Gronwall's Lemma.

Step 2. General case. Let $p^0(t) = 0$. For $n \ge 1$, define $(p^n(t), q^n(t), r^n(t,z))$ to be the unique solution to the following BSDE:

$$\begin{aligned}
dp^n(t) &= E[F(t, p^{n-1}(t), p^{n-1}(t+\delta), p_t^{n-1}, q^n(t), q^n(t+\delta), q_t^n, r^n(t,\cdot), r^n(t+\delta,\cdot), r_t^n(\cdot))|\mathcal{F}_t]dt\\
&\quad + q^n(t)dB_t + r^n(t,z)\tilde{N}(dt,dz),
\end{aligned} \tag{4.25}$$

$$p^n(t) = G(t); \qquad t\in[T, T+\delta].$$

The existence of $(p^n(t), q^n(t), r^n(t,z))$ is proved in Step 1. By the same arguments leading to (4.16), we deduce that

$$\begin{aligned}
&E[|p^{n+1}(t)-p^n(t)|^2]+\frac{1}{2}E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|^2 ds\nu(dz)\right]\\
&\quad+\frac{1}{2}E\left[\int_t^T|q^{n+1}(s)-q^n(s)|^2ds\right]\\
&\leq CE\left[\int_t^T|p^{n+1}(s)-p^n(s)|^2ds\right]+\frac{1}{2}E\left[\int_t^T|p^n(s)-p^{n-1}(s)|^2ds\right] \qquad (4.26)
\end{aligned}$$

This implies that

$$-\frac{d}{dt}\left(e^{Ct}E\left[\int_t^T|p^{n+1}(s)-p^n(s)|^2ds\right]\right)\leq\frac{1}{2}e^{Ct}E\left[\int_t^T|p^n(s)-p^{n-1}(s)|^2ds\right] \qquad (4.27)$$

Integrating (4.27) from u to T we get

$$\begin{aligned}
&E\left[\int_u^T|p^{n+1}(s)-p^n(s)|^2ds\right]\leq\frac{1}{2}\int_u^T dte^{C(t-u)}E\left[\int_t^T|p^n(s)-p^{n-1}(s)|^2ds\right]\\
&\leq e^{CT}\int_u^T dtE[\int_t^T|p^n(s)-p^{n-1}(s)|^2ds]. \qquad (4.28)
\end{aligned}$$

Iterating the above inequality we obtain that

$$E[\int_0^T|p^{n+1}(s)-p^n(s)|^2ds]\leq\frac{e^{CnT}T^n}{n!}$$

Using above inequality and a similar argument as in Step 1, it can be shown that $(p^n(t),q^n(t),r^n(t,z))$ converges to some limit $(p(t),q(t),r(t,z))$, which is the unique solution of equation (4.9). □

Theorem 4.3 *Assume* $E\left[\sup_{T\leq t\leq T+\delta}|G(t)|^{2\alpha}\right]<\infty$ *for some* $\alpha>1$ *and that the following condition hold:*

$$\begin{aligned}
&|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r)-F(t,\bar{p}_1,\bar{p}_2,\bar{p},\bar{q}_1,\bar{q}_2,\bar{q},\bar{r}_1,\bar{r}_2,\bar{r})|\\
&\quad\leq C(|p_1-\bar{p}_1|+|p_2-\bar{p}_2|+\sup_{0\leq s\leq\delta}|p(s)-\bar{p}(s)|+|q_1-\bar{q}_1|+|q_2-\bar{q}_2|+|q-\bar{q}|_{V_1}\\
&\qquad+|r_1-\bar{r}_1|_{\mathcal{H}}+|r_2-\bar{r}_2|_{\mathcal{H}}+|r-\bar{r}|_{V_2}). \qquad (4.29)
\end{aligned}$$

Then the BSDE (4.9) *admits a unique solution* $(p(t),q(t),r(t,z))$ *such that*

$$E\left[\sup_{0\leq t\leq T}|p(t)|^{2\alpha}+\int_0^T\{q^2(t)+\int_{\mathbb{R}}r^2(t,z)\nu(dz)\}dt\right]<\infty.$$

Proof.

Step 1 . Assume F is independent of p_1, p_2 and p. In this case condition (4.29) reduces to assumption (4.1). By the Step 1 in the proof of Theorem 4.2, there is a unique solution $(p(t), q(t), r(t,z))$ to equation (4.9).

Step 2. General case. Let $p^0(t) = 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t,z))$ to be the unique solution to the following BSDE:

$$\begin{aligned} dp^n(t) =& E[F(t, p^{n-1}(t), p^{n-1}(t+\delta), p_t^{n-1}, q^n(t), q^n(t+\delta), q_t^n, r^n(t,\cdot), r^n(t+\delta,\cdot), r_t^n(\cdot))|\mathcal{F}_t]dt \\ &+ q^n(t)dB_t + r^n(t,z)\tilde{N}(dt,dz), \end{aligned} \tag{4.30}$$

$$p^n(t) = G(t), \qquad t \in [T, T+\delta].$$

By Step 1, $(p^n(t), q^n(t), r^n(t,z))$ exists. We are going to show that $(p^n(t), q^n(t), r^n(t,z))$ forms a Cauchy sequence. Using Itô's formula, we have

$$\begin{aligned} &|p^{n+1}(t) - p^n(t)|^2 + \int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|^2 ds\nu(dz) + \int_t^T |q^{n+1}(s) - q^n(s)|^2 ds \\ &\quad = -2\int_t^T (p^{n+1}(s) - p^n(s)) \\ &\times [E[F(s, p^n(s), p^n(s+\delta), p_s^n, q^{n+1}(s), q^{n+1}(s+\delta), q_s^{n+1}, r^{n+1}(s,\cdot), r^{n+1}(s+\delta,\cdot), r_s^{n+1}(\cdot)) \\ &- F(s, p^{n-1}(s), p^{n-1}(s+\delta), p_s^{n-1}, q^n(s), q^n(s+\delta), q_s^n, r^n(s,\cdot), r^n(s+\delta,\cdot), r_s^n(\cdot))|\mathcal{F}_s]]ds \\ &- 2\int_t^T (p^{n+1}(s) - p^n(s))(q^{n+1}(s) - q^n(s))dB_s \\ &- \int_t^T \int_{\mathbb{R}} [|r^{n+1}(s,z) - r^n(s,z)|^2 + 2(p^{n+1}(s-) - p^n(s-))(r^{n+1}(s,z) - r^n(s,z))]\tilde{N}(ds,dz) \end{aligned} \tag{4.31}$$

Take conditional expectation with respect to $\mathcal{F}_t$, take the supremum over the interval $[u,T]$ and use the condition (4.29) to get

$$
\begin{aligned}
&\sup_{u\le t\le T}|p^{n+1}(t)-p^n(t)|^2+\sup_{u\le t\le T}E\left[\int_t^T|q^{n+1}(s)-q^n(s)|^2ds|\mathcal{F}_t\right]\\
&\quad+\sup_{u\le t\le T}E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|^2ds\nu(dz)|\mathcal{F}_t\right]\\
&\le C_\varepsilon\sup_{u\le t\le T}E\left[\int_u^T|p^{n+1}(s)-p^n(s)|^2ds|\mathcal{F}_t\right]\\
&\quad+C_1\varepsilon\sup_{u\le t\le T}E\left[\int_u^T|p^n(s)-p^{n-1}(s)|^2ds|\mathcal{F}_t\right]\\
&\quad+C_2\varepsilon\sup_{u\le t\le T}E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]\\
&\quad+C_3\varepsilon\sup_{u\le t\le T}E\left[\int_t^T|q^{n+1}(s)-q^n(s)|^2ds|\mathcal{F}_t\right]\\
&\quad+C_4\varepsilon\sup_{u\le t\le T}E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|^2ds\;\nu(dz)|\mathcal{F}_t\right]
\end{aligned}
\tag{4.32}
$$

Choosing $\varepsilon>0$ such that $C_3\varepsilon<1$ and $C_4\varepsilon<1$ it follows from (4.32) that

$$
\begin{aligned}
\sup_{u\le t\le T}|p^{n+1}(t)-p^n(t)|^2&\le C_\varepsilon\sup_{u\le t\le T}E\left[\int_u^T|p^{n+1}(s)-p^n(s)|^2ds|\mathcal{F}_t\right]\\
&\quad+(C_1+C_2)\varepsilon\sup_{u\le t\le T}E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]
\end{aligned}
\tag{4.33}
$$

Note that $E\left[\int_u^T|p^{n+1}(s)-p^n(s)|^2ds|\mathcal{F}_t\right]$ and $E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]$ are right-continuous martingales on $[0,T]$ with terminal random variables $\int_u^T|p^{n+1}(s)-p^n(s)|^2ds$ and $\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^2|\mathcal{F}_s]ds$. Thus for $\alpha>1$, we have

$$
\begin{aligned}
&E\left[\left(\sup_{u\le t\le T}E\left[\int_u^T|p^{n+1}(s)-p^n(s)|^2ds|\mathcal{F}_t\right]\right)^\alpha\right]\le c_\alpha E\left[\left(\int_u^T|p^{n+1}(s)-p^n(s)|^2ds\right)^\alpha\right]\\
&\le c_{T,\alpha}E\left[\int_u^T\sup_{s\le v\le T}|p^{n+1}(v)-p^n(v)|^{2\alpha}ds\right],
\end{aligned}
\tag{4.34}
$$

and

$$E\left[\left(\sup_{u\le t\le T}E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]\right)^\alpha\right]$$
$$\le c_{T,\alpha}E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^{2\alpha}|\mathcal{F}_s]ds\right]$$
$$\le c_{T,\alpha}E\left[\int_u^T \sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^{2\alpha}ds\right], \tag{4.35}$$

(4.33), (4.34) and (4.35) yield that for $\alpha > 1$,

$$E\left[\sup_{u\le t\le T}|p^{n+1}(t)-p^n(t)|^{2\alpha}\right]\le C_{1,\alpha}E\left[\int_u^T\sup_{s\le v\le T}|p^{n+1}(v)-p^n(v)|^{2\alpha}ds\right]$$
$$+C_{2,\alpha}E\left[\int_u^T\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|^{2\alpha}ds\right] \tag{4.36}$$

Put

$$g_n(u)=E\left[\int_u^T\sup_{t\le s\le T}|p^n(s)-p^{n-1}(s)|^{2\alpha}\right]$$

(4.36) implies that

$$-\frac{d}{dt}(e^{C_{1,\alpha}u}g_{n+1}(u))\le e^{C_{1,\alpha}u}C_{2,\alpha}g_n(u) \tag{4.37}$$

Integrating (4.37) from t to T we get

$$g_{n+1}(t)\le c_{2,\alpha}\int_t^T e^{C_{1,\alpha}(s-t)}g_n(s)ds\le C_{2,\alpha}e^{C_{1,\alpha}T}\int_t^T g_n(s)ds. \tag{4.38}$$

Iterating the above inequality we obtain that

$$E\left[\int_0^T\sup_{t\le s\le T}|p^{n+1}(s)-p^n(s)|^{2\alpha}dt\right]\le\frac{e^{CnT}T^n}{n!}$$

Using above inequality and a similar argument as in step 1, we can show that $(p^n(t),q^n(t),r^n(t,z))$ converges to some limit $(p(t),q(t),r(t,z))$, which is the unique solution of equation (4.9). □

Finally we present a result when the coefficient f is independent of z and r.

Theorem 4.4 *Assume* $E\left[\sup_{T\le t\le T+\delta}|G(t)|^2\right]<\infty$ *and* F *satisfies*

$$|F(t,y_1,y_2,p)-F(t,\bar y_1,\bar y_2,\bar p)|\le C(|y_1-\bar y_1|+|y_2-\bar y_2|+\sup_{0\le s\le\delta}|p(s)-\bar p(s)|). \tag{4.39}$$

Then the backward stochastic differential equation (4.9) *admits a unique solution.*

Proof. Let $p^0(t) = 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t,z))$ to be the unique solution to the following BSDE:

$$dp^n(t) = E[F(t, p^{n-1}(t), p^{n-1}(t+\delta), p_t^{n-1})|\mathcal{F}_t]dt + q^n(t)dB_t + r^n(t,z)\tilde{N}(dt,dz), \qquad (4.40)$$

$$p^n(t) = G(t) \qquad t \in [T, T+\delta].$$

We will show that $(p^n(t), q^n(t), r^n(t,z))$ forms a Cauchy sequence. Subtracting p^n from p^{n+1} and taking conditional expectation with respect to $\mathcal{F}_t$ we get

$$\begin{aligned} & p^{n+1}(t) - p^n(t) \\ = \quad & -E[\int_t^T (E[F(s, p^n(s), p^n(s+\delta), p_s^n)|\mathcal{F}_s] \\ & \quad -E[F(s, p^{n-1}(s), p^{n-1}(s+\delta), p_s^{n-1})|\mathcal{F}_s])ds|\mathcal{F}_t] \end{aligned} \qquad (4.41)$$

Take the supremum over the interval $[u, T]$ and use the assumption (4.39) to get

$$\begin{aligned} \sup_{u \leq t \leq T} |p^{n+1}(t) - p^n(t)|^2 &\leq C \sup_{u \leq t \leq T} \left(E\left[\int_u^T |p^n(s) - p^{n-1}(s)|ds|\mathcal{F}_t \right] \right)^2 \\ &+ C \sup_{u \leq t \leq T} \left(E\left[\int_u^T E[\sup_{s \leq v \leq T} |p^n(v) - p^{n-1}(v)||\mathcal{F}_s]ds|\mathcal{F}_t \right] \right)^2 \end{aligned} \qquad (4.42)$$

By the Martingale Inequality, we have

$$\begin{aligned} & E\left[\left(\sup_{u \leq t \leq T} E\left[\int_u^T |p^n(s) - p^{n-1}(s)|ds|\mathcal{F}_t \right] \right)^2 \right] \leq cE\left[\left(\int_u^T |p^n(s) - p^{n-1}(s)|ds \right)^2 \right] \\ & \leq c_T E\left[\int_u^T \sup_{s \leq v \leq T} |p^n(v) - p^{n-1}(v)|^2 ds \right], \end{aligned} \qquad (4.43)$$

and

$$\begin{aligned} & E\left[\left(\sup_{u \leq t \leq T} E\left[\int_u^T E[\sup_{s \leq v \leq T} |p^n(v) - p^{n-1}(v)||\mathcal{F}_s]ds|\mathcal{F}_t \right] \right)^2 \right] \\ \leq \quad & c_T E\left[\int_u^T E[\sup_{s \leq v \leq T} |p^n(v) - p^{n-1}(v)|^2|\mathcal{F}_s]ds \right], \end{aligned} \qquad (4.44)$$

Taking expectation on both sides of (4.42) gives

$$E\left[\sup_{u \leq t \leq T} |p^{n+1}(t) - p^n(t)|^2 \right] \leq C \int_u^T E\left[\sup_{s \leq v \leq T} |p^n(v) - p^{n-1}(v)|^2]ds \right] \qquad (4.45)$$

It follows easily from here that $(p^n(t), q^n(t), r^n(t,z))$ converges to some limit $(p(t), q(t), r(t,z))$, which is the unique solution of equation (4.9). $\square$

5 Example

5.1 Optimal consumption from a cash flow with delay

Let $\alpha(t), \beta(t)$ and $\gamma(t,z)$ be given bounded adapted processes, $\alpha(t)$ deterministic. Assume that $\int_{\mathbb{R}} \gamma^2(t,z)\nu(dz) < \infty$. Consider a cash flow $X^0(t)$ with a dynamics

$$dX^0(t) = X^0(t-\delta)\left[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right] \; ; \; t \in [0,T] \tag{5.1}$$

$$X^0(t) = x_0(t) > 0 \; ; \; t \in [-\delta, 0], \tag{5.2}$$

where $x_0(t)$ is a given bounded deterministic function.

Suppose that at time $t \in [0,T]$ we consume at the rate $c(t) \geq 0$, a càdlàg adapted process. Then the dynamics of the corresponding net cash flow $X(t) = X^c(t)$ is

$$dX(t) = [X(t-\delta)\alpha(t) - c(t)]dt + X(t-\delta)\beta(t)dB(t) + X(t-\delta)\int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz) \; ; \; t \in [0,T] \tag{5.3}$$

$$X(t) = x_0(t) \; ; \; t \in [-\delta, 0]. \tag{5.4}$$

Let $U_1(t,c,\omega) : [0,T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ be a given stochastic utility function satisfying the following conditions

$$\begin{aligned} t \to U_1(t,c,\omega) &\text{ is } \mathcal{F}_t\text{-adapted for each } c \geq 0, \\ c \to U_1(t,c,\omega) &\text{ is } \mathcal{C}^1, \; \frac{\partial U_1}{\partial c}(t,c,\omega) > 0, \\ c \to \frac{\partial U_1}{\partial c}&(t,c,\omega) \text{ is strictly decreasing} \\ \lim_{c\to\infty} \frac{\partial U_1}{\partial c}(t,c,\omega) &= 0 \text{ for all } t,\omega \in [0,T] \times \Omega. \end{aligned} \tag{5.5}$$

Put $v_0(t,\omega) = \frac{\partial U_1}{\partial c}(t,0,\omega)$ and define

$$I(t,v,\omega) = \begin{cases} 0 & \text{if } v \geq v_0(t,\omega) \\ \left(\frac{\partial U_1}{\partial c}(t,\cdot,\omega)\right)^{-1}(v) & \text{if } 0 \leq v < v_0(t,\omega) \end{cases} \tag{5.6}$$

Suppose we want to find the consumption rate $\hat{c}(t)$ such that

$$J(\hat{c}) = \sup\{J(c) \; ; \; c \in \mathcal{A}\} \tag{5.7}$$

where

$$J(c) = E\left[\int_0^T U_1(t,c(t),\omega)dt + kX(T)\right].$$

Here $k > 0$ is constant and $\mathcal{A}$ is the family of all càdlàg, $\mathcal{F}_t$-adapted processes $c(t) \geq 0$ such that $E[|X(T)|] < \infty$.

In this case the Hamiltonian given by (1.6) gets the form

$$H(t, x, y, a, u, p, q, r(\cdot), \omega) = U_1(t, c, \omega) + (\alpha(t)y - c)p \\ + y\beta(t)q + y\int_{\mathbb{R}} \gamma(t, z)r(z)\nu(dz). \tag{5.8}$$

Maximizing H with respect to c gives the following first order condition for an optimal $\hat{c}(t)$:

$$\frac{\partial U_1}{\partial c}(t, \hat{c}(t), \omega) = p(t). \tag{5.9}$$

The time-advanced BSDE for $p(t), q(t), r(t, z)$ is, by (1.8)-(1.9)

$$dp(t) = -E[\left\{\alpha(t)p(t+\delta) + \beta(t)q(t+\delta) + \int_{\mathbb{R}} \gamma(t, z)r(t+\delta, z)\nu(dz)\right\}\chi_{[0,T-\delta]}(t)|\mathcal{F}_t]dt \\ + q(t)dB(t) + \int_{\mathbb{R}} r(t, z)\tilde{N}(dt, dz)\ ;\ t \in [0, T] \tag{5.10}$$

$$p(T) = k. \tag{5.11}$$

Since k is deterministic, we can choose $q = r = 0$ and (5.10)-(5.11) becomes

$$dp(t) = -\alpha(t)p(t+\delta)\chi_{[0,T-\delta]}(t)dt\ ;\ t < T \tag{5.12}$$

$$p(t) = k \text{ for } t \in [T-\delta, T+\delta]. \tag{5.13}$$

To solve this we introduce

$$h(t) := p(T-t)\ ;\ t \in [-\delta, T].$$

Then

$$dh(t) = -dp(T-t) = \alpha(T-t)p(T-t+\delta)dt \\ = \alpha(T-t)p(T-(t-\delta))dt = \alpha(T-t)h(t-\delta)dt \tag{5.14}$$

for $t \in [0, T]$, and

$$h(t) = p(T-t) = k \text{ for } t \in [-\delta, 0]. \tag{5.15}$$

This determines $h(t)$ inductively on each interval $[j\delta, (j+1)\delta]\ ;\ j = 1, 2, \ldots,$ as follows:

If $h(s)$ is known on $[(j-1)\delta, j\delta]$, then

$$h(t) = h(j\delta) + \int_0^t h'(s)ds = h(j\delta) + \int_{j\delta}^t \alpha(T-s)h(s-\delta)ds\ ;\ j \in [j\delta, (j+1)\delta]. \tag{5.16}$$

We have proved

Proposition 5.1 *The optimal consumption rate* $\hat{c}_\delta(t)$ *for the problem* (5.3)-(5.4), (5.7) *is given by*

$$\hat{c}_\delta(t) = I(t, h_\delta(T-t), \omega), \tag{5.17}$$

where $h_\delta(\cdot) = h(\cdot)$ *is determined by* (5.15)-(5.16).

Remark 5.2 Assume that $\alpha(t) = \alpha > 0$ *for all* $t \in [0, T]$. *Then we see by induction on* (5.16) *that*

$$0 \le \delta_1 < \delta_2 \Rightarrow h_{\delta_1}(t) > h_{\delta_2(t)} \text{ for all } t \in (0, T]$$

and hence, perhaps suprisingly,

$$0 \le \delta_1 < \delta_2 \Rightarrow \hat{c}_{\delta_1}(t) < \hat{c}_{\delta_2}(t) \text{ for all } t \in [0, T).$$

Thus the optimal consumption rate increases if the delay increases. The explanation for this may be that the delay postpones the negative effect on the growth of the cash flow caused by the consumption.

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